# On the Complete Integral Closure of Rings that Admit a $\phi$-Strongly Prime Ideal

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ABSTRACT:

Let $R$ be a commutative ring with 1 and $T(R)$ be its total quotient ring such that $Nil(R)$ (the set of all nilpotent elements of $R$) is a divided prime ideal of $R$. Then $R$ is called a *$\phi$-chained ring ($\phi$-CR)* if for every $x, y \in R \setminus Nil(R)$, either $x \mid y$ or $y \mid x$. A prime ideal $P$ of $R$ is said to be a $\phi$-strongly prime ideal if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ or $aP \subset bP$. In this paper, we show that if $R$ admits a regular $\phi$-strongly prime ideal, then either $R$ does not admit a minimal regular prime ideal and $c(R)$ (the complete integral closure of $R$ inside $T(R)$) $= T(R)$ is a $\phi$-CR or $R$ admits a minimal regular prime ideal $Q$ and $c(R) = (Q : Q)$ is a $\phi$-CR with maximal ideal $Q$. We also prove that the complete integral closure of a conducive domain is a valuation domain.

## 1 INTRODUCTION

We assume throughout that all rings are commutative with $1 \neq 0$. We begin by recalling some background material. As in [17], an integral domain $R$, with quotient field $K$, is called a ***pseudo-valuation domain (PVD)*** in case each prime ideal $P$ of $R$ is ***strongly prime***, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [4], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [4] that a prime ideal $P$ of $R$ is said to be ***strongly prime (in R)*** if $aP$ and $bR$ are comparable (under inclusion) for all $a, b \in R$. A ring $R$ is called a ***pseudo-valuation ring (PVR)*** if each prime ideal of $R$ is strongly prime. A PVR is necessarily quasilocal [4, Lemma 1(b)]; a chained ring is a PVR [4, Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [6, Proposition 3]). Recall from [7] and [14] that a prime ideal $P$ of $R$ is called ***divided*** if it is comparable (under inclusion) to every ideal of $R$. A ring $R$ is called a ***divided ring*** if every prime ideal of $R$ is divided.

In [8], the author gave another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors). As in [8], for a ring $R$ with total quotient ring $T(R)$ such that $Nil(R)$ (the set of all nilpotent elements of $R$) is a divided prime ideal of $R$, let $\phi : T(R) \longrightarrow K := R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then $\phi$ is a ring homomorphism from $T(R)$ into $K$, and $\phi$ restricted to $R$ is also a ring homomorphism from $R$ into $K$ given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal $Q$ of $\phi(R)$ is called a *K-strongly prime* ideal if $xy \in Q$, $x \in K, y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$ is called a *K-pseudo-valuation ring (K-PVR)*. A prime ideal $P$ of $R$ is called a *$\phi$-strongly prime* ideal if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If a $\phi$-strongly prime ideal $P$ of $R$ contains a nonzerodivisor, then we say that $P$ is a regular $\phi$-strongly prime ideal. If each prime ideal of $R$ is $\phi$-strongly prime, then $R$ is called a *$\phi$-pseudo-valuation ring ($\phi - PVR$)*. For an equivalent characterization of a $\phi$-PVR, see Proposition 1.1(7). It was shown in [9, Theorem 2.6] that for each $n \geq 0$ there is a $\phi$-PVR of Krull dimension $n$ that is not a PVR. Also, recall from [10], that a ring $R$ is called a *$\phi$-chained ring ($\phi$-CR)* if $Nil(R)$ is a divided prime ideal of $R$ and for every $x \in R_{Nil(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$. For an equivalent characterization of a $\phi$-CR, see Proposition 1.1(9). A $\phi$-CR is a divided ring [10, Corollary 3.3(2)], and hence is quasilocal. It was shown in [10, Theorem 2.7] that for each $n \geq 0$ there is a $\phi$-CR of Krull dimension $n$ that is not a chained ring.

Suppose that $Nil(R)$ is a divided prime ideal of a commutative ring $R$ such that $R$ admits a regular $\phi$-strongly prime. In this paper, we show that $c(R)$ (the complete integral closure of $R$ inside $T(R)$) is a $\phi$-chained ring. In fact, we will show that either $c(R) = T(R)$ or $c(R) = (Q : Q) = \{x \in T(R) : xQ \subset Q\}$ for some minimal regular $\phi$-strongly prime ideal $Q$ of $R$.

In the following proposition, we summarize some basic properties of PVRs, $\phi$-PVRs, and $\phi$-CRs.

PROPOSITION 1.1.

1. *An integral domain is a PVR if and only if it is a $\phi$-PVR if and only if it is a PVD( [1, Proposition 3.1], [2, Proposition 4.2], [6, Proposition 3], and [8]).*
2. *A PVR is a divided ring [4, Lemma 1], and hence is quasilocal.*
3. *A ring $R$ is a PVR if and only if for every $a, b \in R$, either $a \mid b$ in $R$ or $b \mid ac$ in $R$ for each nonunit $c$ in $R$ [4, Theorem 5].*
4. *If $R$ is a PVR, then $Nil(R)$ and $Z(R)$ are divided prime ideals of $R$ ([4], [8]).*
5. *A PVR is a $\phi$-PVR [8, Corollary 7(3)].*
6. *If $P$ is a $\phi$-strongly prime ideal of $R$, then $P$ is a divided prime. In particular, if $R$ is a $\phi$-PVR, then $R$ is a divided ring [8, Proposition 4], and hence is quasilocal.*
7. *Suppose that $Nil(R)$ is a divided prime ideal of $R$. Then a prime ideal $P$ of $R$ is $\phi$-strongly prime if and only if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in $R$ or $aP \subset bP$. In particular, a ring $R$ is a $\phi$-PVR if and only if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in $R$ or $b \mid ac$ in $R$ for every nonunit $c \in R$ [8, Corollary 7].*
8. *Suppose that $Nil(R)$ is a divided prime ideal of $R$. If $P$ is a $\phi$-strongly prime ideal of $R$ and $Q$ is a prime ideal of $R$ contained in $P$, then $Q$ is a $\phi$-strongly prime ideal of $R$ [8, Proposition 5].*

9. *Suppose that $Nil(R)$ is a divided prime ideal of $R$. Then a ring $R$ is a $\phi$-CR if and only if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in $R$ or $b \mid a$ in $R$ [10, Proposition 2.3].*

10. *A $\phi$-CR is a $\phi$-PVR [10, Corollary 2.3].* □

# 2 The COMPLETE INTEGRAL CLOSURE OF RINGS THAT ADMIT A REGULAR $\phi$-STRONGLY PRIME IDEAL

Throughout this section, $Nil(R)$ denotes the set of all nilpotent elements of $R$, $Z(R)$ denotes the set of all zerodivisor elements of $R$, and $c(R)$ denotes the complete integral closure of $R$ inside $T(R)$. The following two lemmas are needed in the proof of Proposition 2.3.

LEMMA 2.1. *Suppose $Nil(R)$ is a divided prime ideal of $R$ and $P$ is a regular $\phi$-strongly prime ideal of $R$. If $s$ is a regular element of $R$ and $z \in Z(R)$, then $s \mid z$ in $R$. In particular, $Z(R) \subset P$.*

**Proof:** Let $s$ be a regular element of $P$ and $z \in Z(R)$. Suppose that $s \nmid z$ in $R$. Then $sP \subset zP$ by Proposition 1.1(7). Since $s \in P$, we have $z \mid s^2$ in $R$, which is impossible. Hence, $s \mid z$ in $R$. Thus, $Z(R) \subset P$. Now, suppose that $s$ is a regular element of $R \setminus P$. Since $P$ is divided by Proposition 1.1(6), we conclude that $P \subset (s)$. Hence, since $Z(R) \subset P$, we conclude that $s \mid z$ in $R$. □

LEMMA 2.2. *Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P$ is a regular $\phi$-strongly prime ideal of $R$. Then $x^{-1}P \subset P$ for each $x \in T(R) \setminus R$. In particular, if $x \in T(R) \setminus R$, then $x$ is a unit of $T(R)$.*

**Proof:** First, observe that $Z(R) \subset P$ by Lemma 2.1. Now, let $x = a/b \in T(R) \setminus R$ for some $a \in R$ and for some $b \in R \setminus Z(R)$. Since $b \nmid a$ in $R$, $Z(R) \subset P$, and $P$ is divided, we conclude that $a \in R \setminus Z(R)$. Hence, $x^{-1} \in T(R)$. Thus, since $b \nmid a$ in $R$, we have $bP \subset aP$ by Proposition1.1(7). Thus $x^{-1}P = \frac{b}{a}P \subset P$. □

In light of the Lemmas 2.1 and 2.2, we have the following proposition.

PROPOSITION 2.3. *Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P$ is a regular prime ideal of $R$. Then the following statements are equivalent:*

1. *$P$ is a $\phi$-strongly prime ideal of $R$.*

2. *$(P : P)$ is a $\phi$-CR with maximal ideal $P$.*

**Proof:** **(1) $\Longrightarrow$ (2).** First, we show that $P$ is the maximal ideal of $(P : P)$. Let $s \in R \setminus P$. Then $s$ is a regular element of $R$ (because $P$ is a divided regular prime ideal of $R$, and therefore $Z(R) \subset P$). Hence $1/s \in (P : P)$. Thus, $s$ is a unit of $(P : P)$. Hence, $P$ is the maximal ideal of $(P : P)$. Now, we show that $(P : P)$ is a $\phi$-CR. Since $Nil(R)$ is a divided prime ideal of $R$, $Nil((P : P)) = Nil(R)$. Let $x, y \in (P : P) \setminus Nil(R)$ and suppose that $x \nmid y$ in $(P : P)$. Then $x = a/s$, $y = b/s$

for some $a, b \in R \setminus Nil(R)$, and some $s \in R \setminus Z(R)$. Since $x \nmid y$ in $(P : P)$, it is impossible that $a$ be a regular element of $R$ and $b \in Z(R)$. Thus, we consider three cases. Case 1: suppose that $a \in Z(R)$ and $b \in R \setminus Z(R)$. Then $b \mid a$ in $R$ by Lemma 2.1. Hence, $y \mid x$ in $(P : P)$. Case 2: suppose that $a, b \in R \setminus Z(R)$. Since $x \nmid y$ in $(P : P)$, we conclude that $w = y/x \in T(R) \setminus R$. Hence, $w^{-1}P = \frac{x}{y}P \subset P$ by Lemma 2.2. Hence, $y \mid x$ in $(P : P)$. Case 3: suppose that $a, b \in Z(R)$. Since $x \nmid y$ in $(P : P)$, we conclude that $a \nmid b$ in $R$. Thus, $aP \subset bP$ by Proposition 1.1(7). Let $h$ be a regular element of $P$. Then $ah = bc$ for some $c \in P$. Suppose that $h \mid c$ in $R$. Then $b \mid a$ in $R$. Hence, $y \mid x$ in $(P : P)$. Thus, suppose that $h \nmid c$ in $R$. Then, $c$ is a regular element of $P$. Hence, $f = c/h \in T(R) \setminus R$. Thus, $f^{-1}P = \frac{h}{c}P \subset P$ by Lemma 2.2. Hence, $f^{-1} \in (P : P)$. Thus, $ah = bc$ implies that $xf^{-1} = y$. Hence, $x \mid y$ in $(P : P)$, a contradiction. Thus, $h \mid c$ in $R$, and therefore $y \mid x$ in $(P : P)$. Hence, $(P : P)$ is a $\phi$-CR by Proposition 1.1(9). **(2)** $\Longrightarrow$ **(1)**. This is clear by Proposition 1.1(10). □

PROPOSITION 2.4. *Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P$ is a regular $\phi$-strongly prime ideal of $R$. Then $Q = \cap_{i=1}^{\infty}(s^i)$ is a prime ideal of $R$ for every regular element $s$ of $P$.*

**Proof:** Suppose that $xy \in Q$ for some $x, y \in R$. Since $Z(R) \subset (s^i)$ for each $i \geq 1$ by Lemma 2.1, we conclude that $Z(R) \subset Q$. Hence, we may assume that neither $x \in Z(R)$ nor $y \in Z(R)$. Thus, assume that $x \notin Q$. Then $s^n \nmid x$ for some $n \geq 1$. Hence, $s^nP \subset xP$ by Proposition 1.1(7). In particular, since $s^n \in P$, we have $s^{2n} \subset xP$. Hence, we have $xy \in (s^{2n+i}) \subset xs^iP \subset (xs^i)$ for every $i \geq 1$. Thus, $y \in (s^i)$ for every $i \geq 1$. Hence, $y \in Q$. □

PROPOSITION 2.5. *Let $P$ be a regular prime ideal of $R$. Then $(P : P) \subset c(R)$.*

**Proof:** Let $x \in (P : P)$, and let $s$ be a regular element of $P$. Then $sx^n \in P$ for every $n \geq 1$. Hence, $x$ is an almost integral element of $R$. Thus, $x \in c(R)$. □

PROPOSITION 2.6. *Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P$ is a regular $\phi$-strongly prime ideal of $R$. Then $T(R)$ is a $\phi$-CR.*

**Proof:** First, observe that $Nil(T(R)) = Nil(R)$. Hence, it suffices to show that if $a, b \in R \setminus Nil(R)$, then either $a \mid b$ in $T(R)$ or $b \mid a$ in $T(R)$. Hence, let $a, b \in R \setminus Nil(R)$. Suppose that $a \nmid b$ in $T(R)$. Then $a \nmid b$ in $R$. Hence, $aP \subset bP$ by Proposition 1.1(7). Thus, let $s$ be a regular element of $P$. Then $as = bc$ for some $c \in P$. Thus, $a = b\frac{c}{s}$. Hence, $b \mid a$ in $T(R)$. □

Now, we state our main result in this section

THEOREM 2.7. *Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P$ is a regular $\phi$-strongly prime ideal of $R$. Then exactly one of the following statements must hold:*

1. *$R$ does not admit a minimal regular prime ideal and $c(R) = T(R)$ is a $\phi$-CR.*
2. *$R$ admits a minimal regular prime ideal $Q$ and $c(R) = (Q : Q)$ is a $\phi$-CR with maximal ideal $Q$.*

**Proof:** (1). Suppose that $R$ does not admit a minimal regular prime ideal. We will show that $1/s \in c(R)$ for every regular element $s \in R$. Hence, let $s$ be a regular element of $R$. Suppose that $s \in R \setminus P$. Then $1/s \in (P:P)$ because $P$ is a divided prime ideal of $R$ by Proposition 1.1(6). Hence $1/s \in (P:P) \subset c(R)$ by Proposition 2.5. Thus, suppose that $s \in P$. We will show that there is regular prime ideal $H \subset P$ such that $s \notin H$. Deny. Let $F = \{D : D$ is a regular prime ideal of $R$ and $D \subset P\}$ and $N = \cap_{D \in F} D$. Then, $s \in N$. Now, by Proposition 1.1(8) and (6), we conclude that the prime ideals in the set $F$ are linearly ordered. Hence, $N$ is a minimal regular prime ideal of $R$, which is a contradiction. Thus, there is a regular prime ideal $H \subset P$ such that $s \notin H$. Hence, once again $1/s \in (H:H) \subset c(R)$ by Proposition 2.5. Thus, $c(R) = T(R)$. Now, $T(R)$ is a $\phi$-CR by Proposition 2.6.

(2). Suppose that $Q$ is a minimal regular prime ideal of $R$. First, observe that $Q \subset P$ by Proposition 1.1(6). Thus, $Q$ is a minimal $\phi$-strongly prime ideal of $R$ by Proposition 1.1(8). Now, $(Q:Q) \subset c(R)$ by Proposition 2.5. We will show that $c(R) \subset (Q:Q)$. Suppose there is an $x \in c(R) \setminus R$. Then $x$ is a unit of $T(R)$ by Lemma 2.2. We consider three cases. Case 1: suppose that $x^{-1} \in T(R) \setminus R$. Then $xQ \subset Q$ by Lemma 2.2. Hence, $x \in (Q:Q)$. Case 2: suppose that $x^{-1} \in R \setminus Q$. Then $Q \subset (x^{-1})$ by Proposition 1.1(6). Thus, $x \in (Q:Q)$. Case 3: suppose that $x^{-1} \in Q$. This case can not happen, for if $x^{-1} \in Q$, then $D = \cap_{i=1}^{\infty}(x^{-1})^i$ contains a regular element of $R$ because $x \in c(R)$. But $D$ is a prime ideal of $R$ by Proposition 2.4. Hence, $D$ is a regular prime ideal of $R$ that is properly contained in $Q$. A contradiction, since $Q$ is a minimal regular prime ideal of $R$. Hence, $c(R) = (Q:Q)$. Now, $c(R) = (Q:Q)$ is a $\phi$-CR by Proposition 2.3. □

Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P \neq Nil(R)$ is a $\phi$-strongly prime ideal of $R$. Then observe that $Nil(\phi(R))$ is a divided prime ideal of $\phi(R)$ and $\phi(P)$ is a regular K-strongly prime ideal of $\phi(R)$ (recall that $K = R_{Nil(R)}$). Now, since $\phi(R)_{Nil(\phi(R))} = K_{Nil(R)}$, we may think of $\phi(P)$ as a $\phi$-strongly prime ideal of $\phi(R)$. In light of this argument and Theorem 2.7, we have the following corollary.

COROLLARY 2.8. *Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P \neq Nil(R)$ is a $\phi$-strongly prime ideal of $R$. Then exactly one of the following statements must hold:*

1. *$\phi(R)$ does not admit a minimal regular prime ideal and $c(\phi(R)) = T(\phi(R)) = K_{Nil(R)}$ is a $K$-CR.*
2. *$\phi(R)$ admits a minimal regular prime ideal $Q$ and $c(\phi(R)) = (Q:Q)$ is a $K$-CR.* □

COROLLARY 2.9. *Suppose that $R$ admits a regular strongly prime ideal. Then exactly one of the statements in Theorem 2.7 must hold.* □

COROLLARY 2.10. *Suppose that an integral domain $R$ admits a nonzero strongly prime ideal of $R$. Then exactly one of the statements in Theorem 2.7 must hold (observe that in this case $c(R)$ is a valuation domain).* □

COROLLARY 2.11. *Suppose that $Nil(R)$ is a divided prime ideal of $R$ and $P$ is a regular $\phi$-strongly prime ideal of $R$. If $P$ contains a finite number, say $n$, of regular* *prime ideals of R, $P_1 \subset P_2 \subset \cdots \subset P_{n-1} \subset P_n = P$, then $c(R) = (P_1 : P_1)$.* □

Let $J(R)$ denotes the Jacobson radical ideal of $R$. We have the following result.

COROLLARY 2.12. *Suppose that $R$ is a Prüfer domain such that $J(R)$ contains a nonzero prime ideal of $R$. Then exactly one of the statements in Theorem 2.7 must hold (once again, observe that in this case $c(R)$ is a valuation domain).*

**Proof:** Let $P$ be a nonzero prime ideal of $R$ such that $P \subset J(R)$. Then $P$ is a strongly prime ideal by [11, Proposition 1.3, and the proof of Theorem 4.3]. Hence, the claim is now clear. □

It is well-known [17, Proposition 3.2] that if $R$ is a Noetherian pseudo-valuation domain (which is not a field), then $R$ has Krull dimension one. The following is an alternative proof of this fact.

PROPOSITION 2.13. *([17, Proposition 3.2]). If $R$ is a Noetherian pseudo-valuation domain (which is not a field), then $R$ has Krull dimension one.*

**Proof:** Deny. Let $M$ be the maximal ideal of $R$. Then there is a nonzero prime ideal $P$ of $R$ such that $P \subset M$ and $M \neq P$. Hence, there is an element $m \in M \setminus P$. Since $P$ is divided, we have $P \subset (m)$. Thus, $1/m \in c(R)$. Since $R$ is Noetherian, $1/m$ is also integral over $R$, which is impossible. Hence, $R$ has Krull dimension one. □

# 3 THE COMPLETE INTEGRAL CLOSURE OF CONDUCIVE DOMAINS

Throughout this section, $R$ denotes an integral domain with quotient field $K$, and $c(R)$ denotes the integral closure of $R$ inside $K$. If $I$ is a proper ideal of $R$, then $Rad(I)$ denotes the radical ideal of $R$. Recall from [11], that Houston and the author defined an ideal $I$ of $R$ to be *powerful* if, whenever $xy \in I$ for elements $x, y \in K$, we have $x \in R$ or $y \in R$. Also, recall that in [13, Theorem 4.5] Bastida and Gilmer proved that a domain $R$ shares an ideal with a valuation domain iff each overring of $R$ which is different from the quotient field $K$ of $R$ has a nonzero conductor to $R$. Domains with this property, called *conducive domains*, were explicity defined and studied by Dobbs and Fedder [15], and further studied by Barucci, Dobbs, and Fontana [12] and [16]. In [11, Theorem 4.1], Houston and the author proved the following result.

PROPOSITION 3.1. *([11, Theorem 4.1]) An integral domain $R$ is a conducive domain if and only if $R$ admits a powerful ideal.* □

The following proposition is needed in the proof of Theorem 3.2.

PROPOSITION 3.2. *([11, Theorem 1.5 and Lemma 1.1]). Suppose that $I$ is a proper powerful ideal of $R$. Then $I^2 \subset (s)$ for every $s \in R \setminus Rad(I)$, and $x^{-1}I^2 \subset R$ for every $x \in K \setminus R$.* □

Now, we state the main result of this section.

THEOREM 3.3. *Suppose that $R$ admits a nonzero proper powerful ideal $I$, that is, $R$ is a conducive domain. Then exactly one of the following two statements must hold:*

1. $\cap_{n=1}^{\infty} I^n \neq 0$ *and exactly one of the following two statements must hold:*
   (a) *$R$ does not admit a minimal regular prime ideal and $c(R) = K$ is a valuation domain.*
   (b) *$R$ admits a minimal regular prime ideal $Q$ and $c(R) = (Q : Q)$ is a valuation domain.*
2. $\cap_{n=1}^{\infty} I^n = 0$ *and $c(R) = \{x \in K : x^{-n} \notin Rad(I)$ for every $n \geq 1\}$ is a valuation domain.*

**Proof:** (1). Suppose that $P = \cap_{n=1}^{\infty} I^n \neq 0$. Then $P$ is a nonzero strongly prime ideal of $R$ by [11, Proposition 1.8]. Hence, the claim is now clear by Theorem 2.7.

(2) Suppose that $P = \cap_{n=1}^{\infty} I^n = 0$. Let $S = \{x \in K : x^{-n} \notin Rad(I)$ for every $n \geq 1\}$, and let $x \in c(R)$. We will show that $x \in S$. Since $P = 0$ and $x \in c(R)$, $x^{-n} \notin I$ for every $n \geq 1$. Hence, $x \in S$. Thus, $c(R) \subset S$. Now, let $s \in S$. We will show that $s \in c(R)$. Let $d$ be a nonzero element of $I^2$. Hence, for every $n \geq 1$ we have either $s^{-n} \in K \setminus R$ or $s^{-n} \in R \setminus Rad(I)$. Thus, $ds^n \in R$ for every $n \geq 1$ by Proposition 3.2. Hence, $s \in c(R)$. Thus, $S \subset c(R)$. Therefore, $S = c(R)$. Now, we show that $c(R) = S$ is a valuation domain. Let $x \in K \setminus S$. Then $x^{-n} \in Rad(I)$ for some $n \geq 1$. Hence, $x^n \notin Rad(I)$ for every $n \geq 1$. Thus, $x^{-1} \in S$. Therefore, $c(R) = S$ is a valuation domain. □

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